**CALCULUS OF VARIATIONS AND OPTIMIZATION METHODS**

# Part I. Calculus of variations

We consider different problems of minimization for the different integral functionals. The differential equations are obtained as the necessary conditions of extremum here. The theory is illustrated by physical examples.

## Lecture 3. Euler equation for the Lagrange problem

The problem of the integral functional minimization with given boundary conditions is considered. The unknown function depends from the unique variable. The value under the integral depends from the unknown function and its first derivative. The second order differential Euler equation is obtained as the necessary condition of the optimality. The minimization of length between two points and the fall of the body are considered as examples.

### 3.1. Lagrange problem

We considered before Brachistochrone problem (see Lecture 1). This is the problem of finding a function of one variable with given boundary conditions that minimizes the integral functional. This functional depends from the unknown function and its first derivative. Now we analyze its extension. Consider the functional



where  *F* is a given sufficiently regular function,  is unknown function that satisfies the boundary conditions

 (3.1)

 and  are given numbers. We have the following minimization problem.

**Problem 3.1**. *Find the function v that minimizes the functional I and satisfies the boundary conditions* (3.1).

**Definition 3.1**. *Problem* 3.1 *is called the* ***Lagrange problem****.*

We considered before the minimization problem for the functions. Try to extend that result to our case.

### 3.2. Euler equation

Try to use the previous technique for the analysis the given problem. Let the function *u* be the solution of the Lagrange problem. Then we have the inequality  for all function *v* that satisfies the boundary conditions (3.1). Therefore, we get



If we use the known method (see Lecture 2), then we determine  divide the previous inequality by *h*, and pass to the limit as *h* tends to zero.

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| **Question**: *Why these transformations any not applicable for our case?* |

This method is not applicable, because our argument is a function. Then *h* is a function too. We do not know what we obtain after division of the number  by the function *h* and passing to the limit. Therefore, we correct our transformations.

Definite the function



where *σ* is a number, *h* is a smooth enough function on the interval . We used the analogical technique for the analysis of the minimization problem for the functions of many variables. Then we would like to choose  as the function *v* at the previous inequality.

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| **Question**: *How we can guaranty  that this function vσ satisfies the given boundary conditions?* |

Then function *vσ* satisfies the boundary conditions (3.1) if *h* satisfies the homogeneous boundary conditions (see Figure 3.1)

 (3.2)

**Definition 3.2.** *The function vσ is called the* ***variation of the function*** *u*.



Figure 3.1. The variation of the function.

Note that  Therefore, the functional *I* has the minimum at the point *u* if and only if the number 0 is the point of the minimum for the function *f*.

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| **Conclusion**: *The problem of the functional minimization is transformed  to the problem of minimization for the function of one variable.* |

By the stationary condition, the necessary condition of minimum of the differentiable function is the equality to zero of its derivative. Let the function *F* be differentiable. Denote by  and  the partial derivatives of this function with respect to second and third arguments.

**Lemma 3.1**. *The derivative of the function f at zero point is*

 (3.3)

**Proof**. Define



Using the Taylor formula, we get



where  as  Then



Divide this value by *σ* and pass to the limit as  We have



Using (3.2), we get



Therefore, the equality (3.3) is true.

**Definition 3.3.** *The derivative of the function f at zero is called the* ***variation of the functional*** *I at the point u and denote by* *or more completely* 

Using stationary condition, we obtain the equality

 (3.4)

This is true for all functions *h* that satisfy the boundary conditions (3.2). Therefore, we have the following result.

**Theorem 3.1**. *The variation of the functional is equal to zero at the point of its minimum.*

**Remark 3.1**. This is the analogue of the stationary condition. We determined before (see Lecture 2), that the derivative of the function is equal to zero at the point of its minimum.

We transform the equality (3.4) with using the following result.

**Lemma 3.2 (*Fundamental lemma of calculus of variations*)**. *Let g be a continuous function on the interval*  *and satisfies the equality*

 (3.5)

*for all continuous function h*. *Then the function g* *is equal to zero everywhere.*

**Proof**. Let *y* be an arbitrarily number of the given interval. Choose the function *h* from Figure 3.2, where *ε* is a positive constant.



Figure 3.2. The function *h*.

We have the equality



Using Mean Theorem, we get



where  After dividing by 2*ε* and passing to the limit as  we obtain  This complete the proof of the lemma, because the point *y* is arbitrary.

**Remark 3.2**. We obtain the same result, if the equality (3.5) is true for all functions *h* that satisfy the boundary conditions (3.2).

Using (3.4), (3.5), we get

 (3.6)

By Lemma, 3.2 we obtain

**Theorem 3.2**. *If the smooth enough function u be the solution of the Lagrange problem, then it satisfies the equality*

 (3.7)

*in the interval* .

**Definition 3.3.** *The equation* (3.7) *is called the* ***Euler equation;*** *its solution is called the* ***extremal****.*

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| **Question**: *What kind of equations has the Euler equation?* |

The Euler equation is the second order ordinary differential equation.

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| **Question**: *The solution of the second order differential equation depends from two arbitrary constants. How we can determine it?* |

We add the boundary conditions (3.2) for solving the Euler equation.

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| **Conclusion**: *Lagrange Problem is transformed to the boundary problem* (3.1), (3.7). |

The algorithm of solving of the Lagrange Problem is given in Table 3.1.

Table. 3.1. The algorithm for the analysis of the Lagrange Problem

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| **Step** | **Action** | **Remark** |
| 1 | Definition of the concrete values *F*,,. | The concrete problem is transformed  to the standard form. |
| 2 | Definition  of the Euler Equation. | Calculation of the partial derivatives of *F* and  definition of the concrete equation (3.7). |
| 3 | Finding of the general solution of the Euler Equation. | Finding the general solution of the equation (3.7)  that depends from two arbitrary constants. |
| 4 | Using the boundary conditions. | Definition of the unknown constants  from the equalities (3.1). |
| 5 | Calculation the corresponding value of the functional. | Definition of the corresponding value of *I*. |
| 6 | Analysis of the result. | This result can be no minimum  of the functional. |

### 3.3. Abstract example

**Example 3.1**. *Minimize the functional*



*with the boundary conditions*



We analyze this problem using the known algorithm (see Table 3.1). Determine the parameters of the concrete problem



Determine the Euler equation. We have



Therefore, we obtain the Euler equation



We have non-homogeneous second order differential equation. Its general solution is the sum of the general solution of the corresponding homogeneous and a partial solution of the non-homogeneous equation. The general solution of the equation



is the function



where *c*1 and *c*2 are arbitrary constants. We choose the following solution of the non-homogeneous equation



Then we have the general solution of the Euler equation



Using our boundary conditions, we have



We determine



Thus, we find the solution of the Euler equation with given boundary condition . In reality, our functional can be transformed to the following form



Its value is greater or equal –*π*, besides the value –*π* can be realized for  only. Thus, this is the solution of our problem.

The Euler equation with given boundary conditions has the unique solution now. This is the unique solution of our problem. Therefore, the Euler equation is the necessary and sufficient condition of the functional minimum. However, this property can be false for other examples because we used the stationary condition for obtained the Euler equation, and the stationary condition is necessary condition of minimum only.

For example, we could consider the maximization problem for the given functional. It obvious that the minimization and the maximization problems have the same Euler equation, because we can change the sign before the integral for the transformation the maximization problem to the minimization one. Then the equality  give us the solution of the Euler equation for the maximization problem too. However, this is not the solution of the maximization problem. Therefore, the Euler equation is not sufficient condition here.

### 3.4. The curve with the minimal length

Consider the next example that has the geometric sense.

**Example 3.2.** We have the set of curves on the plane with fixed ends (see Figure 3.3). It is necessary to find the function of this set with minimal length. Give the analytic problem statement.



Figure 3.3. The functions with fixed ends.

Try to determine the length of the curve  Consider a small enough interval  If the value Δ*x* is small enough, then the part of curve on this interval is close enough to the corresponding part of line Δ*s* (see Figure 3.4). Using the Pythagoras theorem, we have the equality  Then we find



Thus, we have the problem of minimization for the functional

 (3.8)

with boundary conditions

 (3.9)

We have the Lagrange Problem with the function

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Figure 3.4. The length of the curve.

Determine Euler equation



We obtain

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Then



Therefore, we have



Hence, the derivative  equal to a constant *c*1. Thus, we have  After integration we have



Use equalities (3.9) for finding two unknown constants. We get



Then we find



Note that the functional (3.8) determine the distance by the curve  between the points determine by equalities (3.9). Our result is trivial because the corresponding curve is the line.

**Remark 3.3**. The problems of maximization of the corresponding functional have the same Euler equations with the same boundary conditions. However, the found solutions minimize the considered functionals. Therefore, it is not solutions of the maximizations problems.

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| **Conclusion**: *the Euler equation is a necessary condition of extremum;  it can be no sufficient condition.* |

**Remark 3.4**. We determined the Euler equation using the stationary condition . It has the necessary condition of minimum only for the arbitrary function *f*. Therefore, the Euler equation is a necessary condition of the extremum too.

### 3.4. The fall of the body

We consider the fall of the body. This phenomenon is described by the height *y.* The general characteristic of the movement is the action *S*(*t*). This is the sum of the kinetic energy *K*(*t*) and the work *W*(*t*) of the given force. Then we get



The work is the product of the force, i.e. the weight *P* and by the height *y*

*U*(*t*) = –*Р* *y*(*t*) = –*mgу*(*t*),

where *m* is the mass of the body, and *g* is the gravitational acceleration. We use the sign “minus” here, because the direction of the force is the contrary of the increasing of the height.

The kinetic energy is proportional to the square of the velocity



Then the mechanical energy of the body in the time *t* is equal to

 (3.10)

If the energy has the constant value *Е*\*, then the energy of the body from the initial time *t*0 to the final time *t*1 is equal to *Е*\*(*t*1 – *t*0). In reality, the energy is variable. Then we can determine the value



with is called the ***action of the system*** on the time interval [*t*0,*t*1]. Let the initial and final height of the body are known. We have boundary conditions

*y*(*t*0) = *y*0, *y*(*t*1) = *y*1. (3.11)

We have the problem of the minimization of the value *I* on the set of the functions *y*, which satisfy the boundary conditions (3.11).

The function *F* is



for this case. Then we obtain Euler equation



Hence we get the known equation of the fall of the body



Thus, the Euler equation give us the classic result of mechanics.

### Outcome

* The Lagrange problem can be transformed to the Euler equation.
* The Euler equation is the necessary condition of extremum for the Lagrange problem.
* The Euler equation is the second order ordinary differential equation.
* The Euler equation is solved with given boundary conditions.
* The Euler equation and the Lagrange problem has the applications in the geometry and mechanics.

### Task. Euler equation for Lagrange problem

Find the function  that minimizes the functional



and satisfies the boundary conditions



The values of the parameters.

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| --- | --- | --- | --- | --- | --- |
| **Variant** |  |  |  |  |  |
| 1 |  | 0 |  | 0 | 1 |
| 2 |  | 0 | 1 | 0 | 1 |
| 3 |  | 0 |  | 0 | -1 |
| 4 |  | 0 | 1 | 0 | 1 |
| 5 |  | 0 |  | 0 | 1 |
| 6 |  | 0 | 1 | 0 | 1 |
| 7 |  | 0 | 1 | 0 | 1 |
| 8 |  | 0 |  | 1 | 0 |
| 9 |  | 0 | 1 | 0 | 1 |
| 10 |  | 0 | *π* | 0 | 1 |
| 11 |  | 0 |  | 0 | 1 |
| 12 |  | 0 |  | 0 | 1 |
| 13 |  | 0 |  | 0 | -1 |
| 14 |  | 0 |  | 0 | -1 |
| 15 |  | 0 |  | 0 | 1 |
| 16 |  | 0 |  | 0 | 1 |
| 17 |  | 0 |  | 1 | 0 |
| 18 |  | 0 |  | 1 | 0 |
| 19 |  | 0 |  | 0 | -1 |
| 20 |  | 0 |  | 0 | -1 |

It is necessary to make the following actions:

1. Give the problem statement.
2. Determine the Euler equation.
3. Find the general solution of the Euler equation that depends from two arbitrary constants.
4. Find these constants by means of the given boundary conditions.
5. Find the corresponding solution of the boundary problem.
6. Calculate the corresponding value of the given functional.
7. Calculate the value of the given functional for the linear function which satisfies the given boundary conditions.
8. Compare these results.

**Remark**. That the general solution of the differential equation  is

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The general solution of the differential equation  is



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### Next step

The problems of minimization for integral functionals have many practical applications. The easiest Lagrange problem can be transformed to the Euler equation. This is a difficult enough second order differential equation. However, sometimes this equation can be simplified. This is the subject of the next lecture.